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**ON THE STABILITY OF A HETEROGENEOUS SHEAR LAYER  
SUBJECT TO A BODY FORCE**

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A handwritten signature in cursive script, reading "Melvin Gerstein".

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Melvin Gerstein, Chief  
*Chemical Physics Section*

**JET PROPULSION LABORATORY**  
California Institute of Technology  
Pasadena, California  
March 25, 1960

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## ABSTRACT

The effects of density variation and body force on the stability of a heterogeneous horizontal shear layer are investigated. The density is assumed to decrease exponentially with height, and the body force is assumed to be derivable from a potential; the velocity distribution in the shear layer is taken to be  $U(y) = \tanh y$ . The method of small disturbances is employed to obtain a family of neutral stability curves depending on the choice of the Richardson number. It is demonstrated, furthermore, that the value of the critical Richardson number depends on the magnitude of the nondimensional density gradient.

## I. INTRODUCTION

It is a frequent occurrence in nature that two fluids of different densities flow one on top of the other. If the flow is predominantly horizontal, and if the density diminishes rapidly upward (e.g., in a mass of air with the temperature increasing with height, such as the "infamous" Los Angeles inversion layer), then the process of turbulent mixing must cause heavier fluid elements to be moved above lighter ones and lighter fluid elements below heavier ones. Both displacements consume energy that has to be extracted from the mean flow at the expense of energy that might be available for the maintenance of turbulence.<sup>a</sup> The same considerations apply quite generally to work against any body force. The present report is one of a series of researches (see, e.g., Refs. 2-6) undertaken to establish the limits of stability of a shear flow in a stably stratified medium.

Whenever one looks into a problem in fluid mechanics, one is bound to discover sooner or later that G. I. Taylor, Prandtl, or von Kármán has already considered the problem and deduced the most important features by simple physical reasoning. The present subject is an exception because both Taylor and Prandtl have not only considered, but have looked into, this problem. In a paper written in 1929 Prandtl (Ref. 2) relates some observations made on "pleasant summer evenings" concerning the stabilizing influence of a density gradient on the wind turbulence near the ground. By an elementary consideration of the balance of forces at play, he arrived at a criterion for stability (in the notation of the present paper):

$$\frac{L}{J} > 1$$

where  $L$  represents a dimensionless density gradient and  $J$  is a Richardson number.

The present, albeit laborious, analysis confirms Prandtl's conjecture that was made, so to say, on the

<sup>a</sup>This argument, which follows Prandtl's exposition (Ref. 1, p. 131), presupposes that the total kinetic energy can be resolved into two terms: one term represents the contribution of the mean flow and the other the turbulent Reynolds stresses. The energy partition is assumed to be unaffected by the density stratification.

back of an old envelope. The physical mechanism underlying the phenomenon, as it happens frequently, is quite simple, and order-of-magnitude estimates can be given provided there is sufficient physical insight into the problem. However, a quantitative description of the phenomenon involves considerably more labor. Such an analysis was made by Taylor (Ref. 3) using the method of small disturbances. It is well known that the use of small, actually infinitesimal, harmonic disturbances permits a linearization of the unsteady equations of motion about the steady state. The steady-state velocity distribution and possibly some of its derivatives appear in the linearized disturbance equation as coefficients. The time-dependent part of the equation can usually split off, leaving an ordinary differential equation. The solution to an ordinary differential equation with non-constant coefficients can ordinarily be given only in terms of an infinite series. For purposes of analytical manipulation in general and stability investigation in particular, a series solution is quite unwieldy. To simplify the coefficients, the velocity profile used by Taylor was made up of straight-line segments yielding solutions in terms of Bessel and Hankel functions, which still proved to be quite difficult to handle. In addition to this difficulty, the presence of the inevitable corners in the velocity profile where the first derivative (i.e., the vorticity!) changes discontinuously has been a source of considerable aggravation. Probably the most important result of Taylor's investigation was

the discovery of a critical value for the Richardson number, say,  $J_{cr}$ , such that for  $J > J_{cr}$  no disturbances that satisfy the imposed boundary conditions are possible.

Drazin (Ref. 5) essentially reworked Taylor's analysis and obtained one of the eigensolutions of the disturbance equation by what Courant so aptly calls the "method of ingenious devices." Drazin limited himself to a very special case:  $L = 0$  and the phase velocity  $c = 0$ . Those limitations have been removed in the present work. The method consists in assuming a simple differentiable velocity profile of the form  $U = \tanh y$ , and a subsequent change in the independent variable from  $y$  to  $U$ . The latter is reminiscent of the hodograph transformation, but it is not quite the same, because  $U(y)$  represents only one part of the velocity vector. By this transformation, the transcendental coefficients of the differential equation introduced by  $U(y)$  are reduced to algebraic ones. Moreover, the domain of interest is shrunk from a doubly infinite one to one that extends from  $-1$  to  $+1$ , a most desirable by-product of this transformation. Next, a change in the dependent variable is performed, patterned after the transformation due to Papperitz (Ref. 7) on the hypergeometric equation. In this form the equation admits of one trivial solution contingent upon certain constraints imposed on  $J$ ,  $L$ ,  $k$ , and  $c$ . Those relations are precisely the ones required to define a neutral stability boundary.

## II. ANALYSIS

The equations of motion governing the behavior of an incompressible inviscid fluid under the action of a body force,  $g \nabla y$ , are Euler's equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} - g \nabla y \quad (1)$$

the condition of incompressibility

$$\frac{D\rho}{Dt} = 0 \quad (2)$$

and the equation of continuity

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

The velocity components, the pressure, and the density are assumed to consist of a time-independent part and a perturbation:

$$u = V[U(y) + \psi'_y(x, y, t)]; v = -V\psi'_x(x, y, t) \quad (4a)$$

$$p = P(y) + p'(x, y, t) \quad (4b)$$

$$\rho = \rho_0 [\bar{p}(y) + \rho'(x, y, t)] \quad (4c)$$

$$\psi' = \psi(y) \exp[ik(x - ct)] \quad (c = c_r + ic_i) \quad (4d)$$

Here,  $V = U(\infty) = -U(-\infty)$ , and  $\psi'$  is a perturbation stream function;  $U(y)$ ,  $P(y)$ , and  $\bar{p}(y)$  describe the ambient state whose stability is to be investigated. Also,  $x = x_1/d$  and  $y = y_1/d$ , where  $x_1$  and  $y_1$  are the physical coordinates and  $d$  is so chosen that  $dU/dy = 1$  at  $y = 0$ ; thus  $d$  characterizes the width of the transition layer. Denoting the Froude number  $V^2/gd$  by  $F$  and eliminating  $\rho'$  and  $p'$  from Eqs. (1), (2), and (3) yields

$$(U - c)(\psi'' - k^2\psi) - U''\psi + (\ln \bar{p})'[(U - c)\psi' - (U - c)'\psi] - \frac{(\ln \bar{p})'}{F} \frac{\psi}{U - c} = 0 \quad (5)$$

where primes denote differentiation with respect to  $y$ . We now set  $\bar{p} = \exp(-2Ly)$  and obtain:

$$(U - c)(\psi'' - k^2\psi) - U''\psi - 2L[(U - c)\psi' - (U - c)'\psi] + \frac{2L}{F} \frac{\psi}{U - c} = 0 \quad (6)$$

The primary velocity distribution is represented by  $U(y) = \tanh y$ , and  $U$  is introduced as the independent variable. We note that  $2L/F = J$  and

$$\frac{dU}{dy} = \text{sech}^2 y = 1 - U^2$$

$$\frac{d^2U}{dy^2} = -2U(1 - U^2)$$

Denoting by primes differentiation with respect to  $U$  and setting  $\psi(y) = \phi(U)$ , we obtain:

$$\phi'' = a(U)\phi' + b(U)\phi = 0 \quad (7)$$

where

$$a(U) = \frac{2(U + L)}{(U + 1)(U - 1)}$$

$$b(U) = \frac{J}{(U - c)^2(U + 1)^2(U - 1)^2} - \frac{k^2}{(U + 1)^2(U - 1)^2} - \frac{2(U + L)}{(U - c)(U + 1)(U - 1)}$$

with the boundary conditions  $k\phi(U) = 0$  at  $U = \pm 1$ .

Equation (7) is of a rather simple type. Its singularities, which are located at  $\pm 1$  and  $c$ , are regular singularities. It can be demonstrated that the point at infinity is also a regular singularity. The substitution into Eq. (7) of

$$Z = (U - 1)^{-\alpha_1}(U - c)^{-\alpha_2}(U + 1)^{-\alpha_3}\phi$$

where  $\alpha_i$  is one of the indices relative to the finite points of the singularity, yields an equation that has at least one bounded solution at each of the singularities. It is of the form

$$Z'' + \left[ \frac{1 - \alpha_1}{U - 1} + \frac{1 - \alpha_2}{U - c} + \frac{1 - \alpha_3}{U + 1} \right] Z' + \frac{(\sigma\tau U - r)}{(U - 1)(U - c)(U + 1)} Z = 0 \quad (8)$$

with the boundary conditions replaced by regularity conditions on  $Z(U)$  at  $U = \pm 1$ , and  $c$ . A sufficient condition for the existence of a trivial solution  $Z = \text{constant}$  (which, of course, satisfies the boundary conditions identically) is given formally by

$$\sigma\tau = 0 \quad (9)$$

$$r = 0 \quad (10)$$

After a considerable amount of tedious but essentially straightforward algebraic manipulation, we obtain from Eqs. (9) and (10) the explicit relations:

$$L = -c \left[ 1 - \frac{4J}{(1-c^2)^2} \right] - \frac{1}{4}(1-\mu) \times [R(1-c)(1-\xi)^{1/2} - R(1+c)(1-\eta)^{1/2} - 2c] \quad (11)$$

$$R^2 = \frac{R(\mu-2) [(1-\xi)^{1/2} (1-\eta)^{1/2}] + J(1-c^2)^{-2} + 2(\mu+1)}{1 + [(1-\xi)(1-\eta)]^{1/2}} \quad (12)$$

where

$$\begin{aligned} R^2 &= L^2 + k^2 \\ \xi &= \frac{2J}{R^2(1-c)^2} \\ \eta &= \frac{2J}{R^2(1+c)^2} \\ \mu &= \left[ 1 - \frac{8J}{(1-c^2)^2} \right]^{1/2} \end{aligned}$$

It is to be noted that Eqs. (11) and (12) are *not* homogeneous in any of the quantities  $J$ ,  $L$ ,  $k$ , and  $c$ ; thus it is always possible to obtain a unique solution. However, there are two equations in four unknowns. It was found convenient from a computational point of view to consider  $L$  and  $R^2$  as the primary variables and  $J$  and  $c$  as parameters. The equations were solved numerically by an iteration technique on an IBM 704 digital computer.<sup>b</sup>

<sup>b</sup>The author is indebted to Dr. P. Peabody for help in programming the problem for machine calculation and to Mr. R. J. Mueller and Mrs. M. Simes who carried out most of the calculations.

### III. DISCUSSION AND RESULTS

In the usual case the neutral stability curve represents possible neutral disturbances and separates the stable from the unstable ones, with no "forbidden" disturbances present anywhere.<sup>c</sup> The stability boundaries displayed in Fig. 1 have a meaning slightly different from that commonly accepted. A particular boundary separates unstable disturbances from stable ones *and* from those that are physically not realizable. This may be stated in a different way: as the boundary is approached from the inside along an arbitrary path one passes over possible disturbances, and the closer one gets to the boundary the smaller will be the amplification. On the other hand, when the boundary is approached from the outside one cannot be sure whether an arbitrary path consists only of permitted disturbances or not. That implies that the only statement that can be made about the attenuation is that if the path consisted only of a succession of possible disturbances, then as the boundary is approached the attenuation decreases to vanish at the boundary itself. For practical purposes, however, this distinction is immaterial since one may still state without

ambiguity the maximum value of  $k^2 + L^2$  which corresponds to an unstable disturbance, for a prescribed value of  $J$ .

The present analysis throws into relief the dependence of the stability on the relative magnitudes of the wave number  $k$  and the parameter  $L$ . The governing quantity appears to be  $k^2 + L^2$ . The analysis of Taylor (Ref. 3) in which the parameter  $L$  was neglected indicated that as the wave number decreased, the instability became more pronounced. By including the effect of the density gradient, one can stabilize a disturbance that was shown, in the absence of a density gradient, to be unstable by virtue of its small wave number. The stabilizing effect of the density gradient is thus clearly demonstrated.

The relationship that must exist between  $L$  and  $c$ , for given values of  $J$ , to obtain a neutral disturbance is displayed in Fig. 2. It is to be noted that the curves terminate before reaching the  $c$ -axis, thus implying that the only solution corresponding to  $L = 0$  is  $c = 0$ , which is the one found by Drazin.

The accepted convention of referring to a *critical* Richardson number is unfortunate because it conjures up similarities with the critical Reynolds number of hydro-

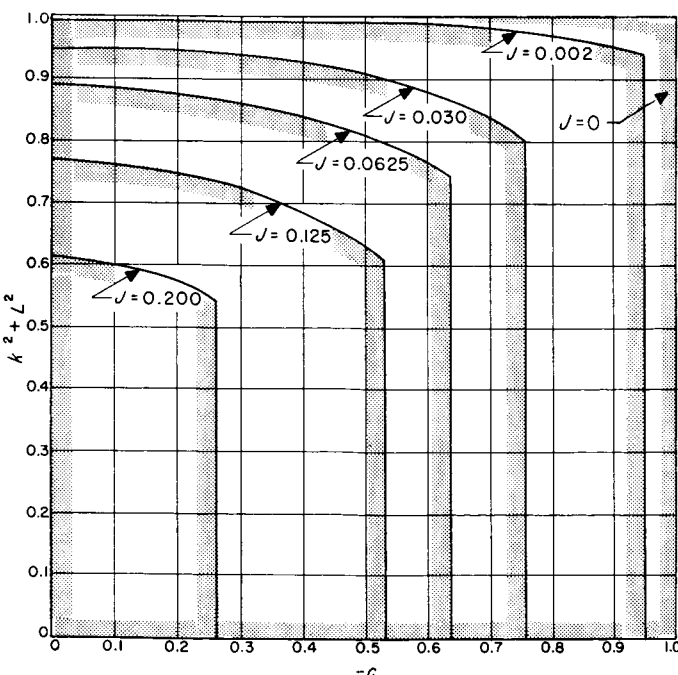


Fig. 1. Stability Boundary

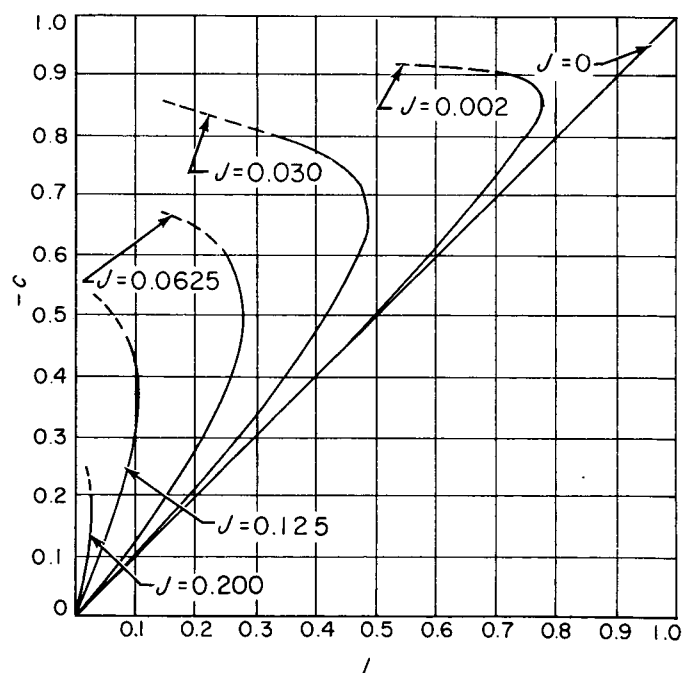


Fig. 2. Neutral Stability Boundary

<sup>c</sup>The existence of "forbidden" disturbances was discovered by Taylor (Ref. 3).

dynamic stability theory to which it bears hardly any semblance. The critical Richardson number represents a number beyond which no virtual displacement<sup>d</sup> of the flow field appears possible. In this sense, then, the flow

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<sup>d</sup>The displacement is to be taken to apply not only to spatial displacements but to velocity, pressure, etc.

is stable, so to say, by default. This is certainly an odd result, but one that has been obtained consistently by Taylor, Goldstein, Drazin, and the present author. No physical explanation can be offered, and the mathematical one provides little solace; it only states that for  $J > J_{cr}$  a physical quantity becomes imaginary when subjected to a virtual displacement.

## NOMENCLATURE

$a(U)$	} functions of $U$ determined by Eq. (7)
$b(U)$	
$c$	$= c_r + ic_i =$ complex phase velocity
$d$	width of transition layer
$F$	Froude number $= V^2/gd$
$g$	gravitational acceleration
$J$	Richardson number
$J_{cr}$	critical value of Richardson number
$k$	wave number
$L$	dimensionless density gradient
$p'$	perturbation pressure
$P$	mean pressure
$r$	accessory parameter (for a discussion of this, see Ref. 8)
$\alpha_i$	one of the indices relative to the finite points of singularity ( $i = 1, 2, 3$ )
$\rho'$	perturbation density
$\bar{\rho}$	mean density
$\rho_0$	reference density
$\sigma$	} exponents relative to the point at infinity
$\tau$	
$\psi'$	perturbation stream function

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